At each point of space the principal curvatures correspond to three directions, mutually perpendicular to one another. When the curves tangent to these directions are the curves of intersection of a triply-orthogonal system of surfaces, the space is called *normal* by Bianchi. All the spaces referred to above are normal. For the cases (7) and (8) the tangents to the curves of intersection x_i = const., x_j = const. are the principal directions.

For the cases of § 4 we put $x_2 = e^{\bar{x}_2 + x_3}$. Then for (11), the curves of intersection of surfaces x_1 = const., $\bar{x}_2$ = const., x_3 = const. have the principal directions. For (14) and the case $k \neq 1$, the principal directions are given by x_1 = const., x_3 = const., and the orthogonal systems of curves on $\bar{x}_2$ = const. defined by

$$\left(\alpha' - \frac{\alpha\gamma'}{\gamma}\right)dx_1^2 + 2\left[\alpha + \frac{1}{4}\alpha'\gamma' - \gamma\left(\frac{\alpha''}{2} - \frac{\alpha'^2}{4\alpha}\right)\right]dx_1\,dx_3 - \gamma\left(\alpha' - \frac{\alpha\gamma'}{\gamma}\right)dx_3^2 = 0.$$

[1] *Mem. Soc. Ital.*, 1896, p. 347.

[2] Levi-Civita, *Rend Lincei* (ser. 5), **26**, 1917, sem. 1 (460).

[3] Bianchi, *Lezioni*, **1**, 377; Cotton, *Ann. Fac. Toul.* (ser. 2), **1**, 1899 (410).

[4] *Science*, **54**, 1921 (305).

[5] *Rend. Lincei* (ser. 5), **27**, 1918, sem. 2 (350).

[6] *Lezioni*, **1**, 354.

GEOMETRIC ASPECTS OF THE ABELIAN MODULAR FUNCTIONS OF GENUS FOUR (II)

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8. *The form* $\binom{3\ 2\ 2}{1\ 1\ 1}$.—This form, written symbolically as $(\rho z)\,(rx)(sy)$, where z is a point in S_3, x a point in S_2, and y a point in S_2', has 36 coefficients and therefore $35-15-8-8 = 4$ absolute projective constants. Points x, y determine a plane which becomes indeterminate for six pairs $x, y = p_i\, q_i$; $(i = 1, \ldots, 6)$ which form associated six points. They are the double singular points of a Cremona transformation T of the fifth order between the planes S_x, S_y. A given plane u is determined by ∞^1 pairs x, y which lie, respectively, on the cubic curves, $(\rho\rho'\rho''u)\,(rx)\,(r'x)\,(r''x)\,(ss's'') = 0$, $(\rho\rho'\rho''u)\,(rr'r'')\,(sy)\,(s'y)\,(s''y) = 0$ These curves pass, respectively, through the six points p_i and the six points q_i. Thus the given form is associated with a general cubic surface, $(\rho z)\,(\rho' z)\,(\rho'' z)\,(rr'r'')\,(ss's'') = 0$, with an isolated double-six of lines and separated

line-sixes. The mapping of the surface from the planes S_x and S_y is given by the above systems of cubics.

9. *The form* $\left(\begin{smallmatrix}1&1&2&2\\1&1&1&1\end{smallmatrix}\right)$.—In the space figure just described we insert a quadric Q with generators t, τ. The point coördinates z can be replaced by bilinear expressions in t, τ and the form $\left(\begin{smallmatrix}3&2&2\\1&1&1\end{smallmatrix}\right)$ becomes a form $(p\tau)$ (πt) (rx) (sy) general of its type, with thirteen absolute constants corresponding to the four absolute constants of the above cubic surface and the nine additional constants for the inserted quadric. The quadric meets the cubic surface in a normal curve of genus 4 so that we have in space the figure of the normal sextic and a particular one of the ∞^4 cubic surfaces through it. In the planes S_x, S_y we have projectively general (with thirteen absolute constants) sextics of genus 4 with nodes at p_i and q_i, respectively, transforms of each other under T. The canonical adjoints of these sextics, with isolated g_1^{3}'s, are obtained by substituting for u in 8 the proper bilinear expressions in t, τ; while a similar substitution for z in the equation of the cubic surface gives the equation on the quadric of the normal curve. We may therefore regard the general form $\left(\begin{smallmatrix}1&1&2&2\\1&1&1&1\end{smallmatrix}\right)$ as a definition of the projectively general plane sextic of genus 4.

10. *The forms* $F, \bar{F}$ *on the conic* $K(\tau)$. *Counter sextics.*—We mark the conic $K(\tau)$ on the plane S_x and plot with reference to it the two counter sextics $\bar{S}_2(t)$ and $S_2(t)$. For each there is a cusp locus of perspective cubics $\bar{G}C(\tau)$ and $GC(\tau)$. Similarly we mark on a plane S_y the conic $K(t)$ and plot with reference to it the two counter sectics $\bar{S}_1(\tau)$, $S_1(\tau)$ which are paired with the above counter sextics, and which have for cusp loci of perspective cubics the sextic curves $GC(t)$, $\bar{G}C(t)$, respectively.

We now consider the form $F = (\alpha t)^3\ (a\tau)^3 = 0$ where τ is a tangent of $K(\tau)$. For variable t we have ∞^1 triangles circumscribed about $K(\tau)$ whose vertices run over a sextic curve. If t determines τ_1, τ_2, τ_3 then the point t, τ_1, τ_2, of this sextic curve is birationally related to the solution t, τ_3 of $F = 0$. Hence the sextic curve is birationally equivalent to $GC(\tau)$ and as a result of the algebraic discussion of the next two sections we prove that this sextic curve is actually $GC(\tau)$. This amounts to the effective elimination of t from $(a\tau_1)^3\ (\alpha t)^3 = 0$, $(a\tau_2)^3\ (\alpha t)^3 = 0$ and the separation of the factor $(\tau_1\tau_2)^3$. Hence we have on the sextic $GC(\tau)$ two g_1^{3}'s such that the ∞^1 t-triads are cusp triangles of perspective cubics of $S_2(t)$ and the ∞^1 τ-triads are triangles circumscribed to $K(\tau)$ for which the intersection of the lines joining each vertex to the contact of the opposite side is the point t of $\bar{S}_2(t)$. Of course a similar statement is true of any one of the four cusp loci.

If we polarize the form F into $(a\tau)$ $(a\tau_1)$ $(a\tau_2)$ (αt) (αt_1) (αt_2) and replace the pairs τ_1, τ_2; t_1, t_2 by points x; y referred in Darboux coördinates to the conics $K(\tau)$; $K(t)$ in Sx; Sy, respectively, we have a form (πx) $(d\tau)$ (δt) (ρy) of the type discussed in 9. Hence the sextics $GC(\tau)$ and $GC(t)$ are transforms of each other under the quintic Cremona transformation T but on the two the role of the t and τ triads with reference to the rational sextics and conics K is reversed. It may be shown that the projective peculiarity (equivalent to four conditions) of our birationally general sextics $\bar{G}C(\tau)$ and $GC(\tau)$ is that their τ triads envelop a conic $K(\tau)$. This same property belongs to the t-triads after transformation of $GC(\tau)$ by T (or $\bar{G}C(\tau)$ by the corresponding $\bar{T}$) into a sextic in the plane S_y.

We have mentioned in I 7 certain intersectional properties of the conic $\bar{b}$, the rational sextic $\bar{S}_2(t)$, and the cusp locus $\bar{G}C(\tau)$. Analogous results for the conic $K(\tau)$, the sextic $\bar{S}_2(t)$, and the locus $GC(\tau)$ are as follows. The conic $K(\tau)$ meets $\bar{S}_2(t)$ in 12 points whose parameters t on $\bar{S}_2(t)$ are the branch points of the function $\tau(t)$ determined by $F = 0$, and whose parameters τ on $K(\tau)$ are the branch point values of this function. The 12 residual function values τ furnish 12 common tangents of $K(\tau)$ and $GC(\tau)$, i.e., the tangents to $GC(\tau)$ at the 12 coincidences of $g_1^3(\tau)$ upon it touch $K(\tau)$. The remaining common tangents of $K(\tau)$ and $GC(\tau)$ are double tangents of $GC(\tau)$ whose parameters on $K(\tau)$ are the branch points of the function $t(\tau)$ in $F = 0$. The two further tangents to $K(\tau)$ from the contacts with $GC(\tau)$ of each double tangent meet in a point which is a coincidence of $g_1^3(t)$ on $GC(\tau)$.

11. *The reciprocity between the forms F, $\bar{F}$; algebraic discussion.*—We denote by A the determinant of the coefficients (without binomial factors) of the form F; the coefficients of F are then, to within numerical factors, three-row minors of A. The reciprocity between the forms F, $\bar{F}$ is then brought out by the fact that the covariant $\bar{F}$ formed for $\bar{F}$ as a ground form is again $F.A^2/3^4$. The invariant $(aA)^3$ $(\alpha\mathrm{A})^3$ is 4 A. This reciprocity fails when A vanishes. Then the four cubics in t furnished by F are linearly dependent. When expressed in terms of three the coefficients in F are three cubics in τ, and the form $F = 0$ can be interpreted as the incidence condition of point τ on a rational *plane* cubic curve C_1 with line t of a rational *plane* cubic envelope C_2. In this case the form $\bar{F}$ factors into two cubics $(A\tau)^3.(\mathrm{A}t)^3$, the conjugate cubics of the rational curves C_1, C_2. Now all the (usually non-vanishing) odd transvectants of the double form $\bar{F}$ will vanish whence in the general case they contain the factor A and are of the second degree for F. From their degree and orders they can be identified at once with $^2/_9$ $(b\tau)^4$ $(\beta t)^4 \equiv (aa')$ $(\alpha\alpha')$ $(a\tau)^2$ $(a'\tau)^2$ $(\alpha t)^2$ $(\alpha' t)^2$; $8\,\delta \equiv (aa')^3$ $(\alpha\alpha')^3$; $^4/_3$ $(\gamma t)^4 \equiv (aa')^3$ $(\alpha\alpha')$ $(\alpha t)^2$ $(\alpha' t)^2$; $^4/_3(c\tau)^4 = (\alpha\alpha')^3$ (aa') $(a\tau)^2$ $(a'\tau)^2$. With respect to the original space cubics C_1, C_2 these forms are interpreted as follows. The form $(b\tau)^4$ $(\beta t)^4$ vanishes when tangent τ of C_1 meets tangent t of C_2; $\delta = 0$ if the null systems of C_1, C_2 are apolar; $(\gamma t)^4$ or $(c\tau)^4$ vanishes if the tangent t of C_2 or the tangent τ of C_1 is a line of the null system of the other curve.

The reciprocity between the dual forms F, $\bar{F}$ is due algebraically to the fact that their coefficients are, respectively, the one- and three- row minors of A. The four comitants of the second degree above are evidently self-dual in meaning and therefore should involve the two-row minors of A. Indeed we find that the 36 coefficients of the four comitants $(b\tau)^4$ $(\beta t)^4$, $(\gamma t)^4$, $(c\tau)^4$, δ are linearly independent in the 36 two-row minors of A. The only remaining comitants of the second degree, not linear in the minors of A, are the two sextics $S_1(\tau)$, $\bar{S}_2(t)$.

The 36 two-row minors of a four-row determinant A, though linearly independent, must satisfy a system of quadratic relations. It is not hard to see that there are precisely 41 such quadratic relations. If we take the above four comitants and form all of *their* comitants of total second degree in the coefficients of the four we have a set of comitants with 666 coefficients which is the number of quadratic combinations of the 36 minors. Hence the coefficients of this new set of comitants must be connected by a system of 41 linear relations due to the existence of the 41 quadratic relations among the minors. These relations will be expressed by the existence of a system of syzygies of the second degree in the coefficients of the four comitants. We find that this system of syzygies is $(b\tau)^4\ (b'\tau)^4\ (\beta\beta')^4 = 6[(c\tau)^4]^2$; $(bb')^4\ (\beta t)^4\ (\beta' t)^4 = 6[(\gamma t)^4]^2$; $(bb')^2\ (b\tau)^2\ (b'\tau)^2\ (\beta\beta')^4 = 6(cc')^2\ (c\tau)^2\ (c'\tau)^2$; $(bb')^4\ (\beta\beta')^2\ (\beta t)^2\ (\beta' t)^2 = 6(\gamma\gamma')^2\ (\gamma t)^2\ (\gamma' t)^2$; $(b\tau)^4\ (\beta\gamma)^4 = 6\,\delta.(c\tau)^4$; $(bc)^4\ (\beta t)^4 = 6\,\delta.(\gamma t)^4$; $(bb')^4\ (\beta\beta')^4 =$

$$6(\gamma\gamma')^4 = 6(cc')^4 = 36\,\delta^2.$$

The 41 coefficients of these syzygies furnish the quadratic relations. The syzygies themselves determine for given $(b\tau)^4\ (\beta t)^4$ the three remaining forms $(c\tau)^4$, $(\gamma t)^4$, δ to within a change of sign of any two.

12. *The fundamental combinants of F, $\bar{F}$.* We have had occasion at times to consider such pencils of cubics in t as are determined by the members $(a\tau_1)^3, (\alpha t)^3, (a\tau_2)^3\ (\alpha t)^3\ (\tau_1 \neq \tau_2)$. If t_1, t_2 belong to the same member of this pencil then the fundamental combinant of Gordan for the pencil is

$$\Gamma = \begin{vmatrix} (a\tau_1)^3\ (\alpha t_1)^3 & (a\tau_2)^3\ (\alpha t_1)^3 \\ (a'\tau_1)^3\ (\alpha' t_2)^3 & (a'\tau_2)^3\ (\alpha' t_2)^3 \end{vmatrix}.$$

Evidently it expresses also that τ_1, τ_2 belong to the same member of the pencil determined by the members $(a\tau)^3\ (\alpha t_1)^3\ (a\tau)^3\ (\alpha t_2)^3\ (t_1 \neq t_2)$. We call Γ the fundamental combinant of F, and $\bar{\Gamma}$ the corresponding fundamental combinant of $\bar{F}$. Each is expressible in terms of the two-row minors of A and therefore in terms of the four comitants of 11. We find that Γ, $\bar{\Gamma} = (\tau_1\tau_2).(t_1t_2) \left\{ (b\tau_1)^2\ (b\tau_2)^2\ (\beta t_1)^2\ (\beta t_2)^2 \pm (t_1t_2)^2.(c\tau_1)^2\ (c\tau_2)^2 \pm (\tau_1\tau_2)^2. (\gamma t_1)^2\ (\gamma t_2)^2 + (t_1t_2)^2.(\tau_1\tau_2)^2.\delta \right\}$

where the upper sign holds for Γ and the lower sign (together with the factor $A/9$) for $\bar{\Gamma}$. Let us call the four terms within this brace, taken with plus signs, K, L, M, N, respectively. Then the involution of binary cubics determined in 7 by the form $(\pi x)\ (d\tau)\ (\delta t)^3$ for variable τ has for fundamental combinant (when x is replaced by τ_1, τ_2) $K + L - M - N$

and the corresponding combinant for the counter sextic $S_2(t)$ rather than $\bar{S}_2(t)$ is $K-L+M-N$. These four involutions all arise from any one by changing the signs of any two of the comitants $(\gamma t)^4$, $(c\tau)^4$, δ. In addition to the facts derived in 10 from this set of involutions other results are easily read off from their simple form. For example the conic $\bar{b}$ defined in 7 for $\bar{S}_2(t)$ and the corresponding conic b for $S_2(t)$ are in a pencil with the conic $K(\tau)$ and meet $K(\tau)$ in the four points determined by $(c\tau)^4 = 0$.